Teaching through Problem Solving

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TtPS is an approach to teaching mathematics that provides students with a way to learn mathematics with understanding.

The patterns studied by mathematicians are, for all practical purposes, as real as the atomic particles studied by physicists.
—Mathematical Sciences Education Board

Teaching through Problem Solving (TtPS) is an effective way to teach mathematics for understanding. It also provides students with a way to learn mathematics with understanding. In this article, we present a definition of what it means to teach through problem solving. We also describe a professional development vignette that exemplifies the teaching moves discussed here. Finally, we reflect about TtPS as a way to achieve equity in the mathematics classroom.

WHAT IS TEACHING THROUGH PROBLEM SOLVING (TTPS)?
For the purposes of this article, we define TtPS as pedagogy that engages students in problem solving as a tool to facilitate students’ learning of important mathematics subject matter and mathematical practices. To illustrate, we provide a sequence of teaching moves that encapsulates the definition and the use of TtPS. However, we first want to share with readers our ultimate agenda: We argue that equitable pedagogy (NCTM 2000) empowers students to be able to make decisions about their life trajectory.

We argue that mathematical empowerment (a component of educational empowerment) provides opportunities, space, and support to students to develop competencies and dispositions so that they might capitalize on and maximize their learning to make life choices. One such empowering tool is fluency in mathematics and mathematical thinking (Mason, Burton, and Stacey 1985; Pólya 1945; Schoenfeld 1985). TtPS provides the space for students to develop such mathematical flexibility. At a local level, TtPS allows all students to enter into the mathematics at hand and enables them to travel through a progressively formalized trajectory that lays out and makes public mathematical practices (solving problems, abstracting, inventing, proving) (Mathematical Sciences Education Board 1990; Romberg 1983).
This article revisits the idea of equity through TtPS in mathematics education and translates it into immediately usable practice. The fundamental notion is that students come to learn mathematics with understanding through problem solving (Hiebert, Carpenter, et al. 1997; Hiebert and Wearne 2003; Mason, Burton, and Stacey 1985; NCTM 2000; Pólya 1945; Schoen and Charles 2003; Schoenfeld 1985; Hiebert, Gallimore, et al. 2003).

TEACHING MOVES

The use of tasks within pedagogies of sense making and problem solving can adhere to the following general pedagogical moves. The roles of teacher and student are embedded in the teaching moves. What follows is a brief account of these teaching moves; a fuller account is presented in the appendix at www.nctm/mt012.

Move 1
Pose the whole task; facilitate students’ understanding of the problem. A caveat: Do not remove the mathematical complexities of the problem. Do not break down the problem so that students will not have to struggle productively in generating understanding.

Move 2
Allow time for students to explore the messiness of the problem, generate conjectures, and build understanding. When the students share their work, the teacher should consider sequencing students’ work, starting from the intuitive-concrete approaches and strategies and progressing to the more abstract approaches. The starting point matters, in the lens of equity, because a starting point that allows all students to enter into the discussion of the mathematics affords students better opportunities to engage with the mathematics and its progression. This notion of progressive formalization (Bransford, Brown, and Cocking 2000) is a most important component of TtPS.

Move 3
Focus on the big idea of the mathematics. Students and teacher should make the big ideas explicit and public and reason about them and from them.

Move 4
Make ideas visible by capturing and recording individual and group contributions for public examination. Focus on using mistakes to reveal difficulties, misconceptions, and reasoning. Support interactivity that encourages students to step out of their comfort zones. Students should realize that making mistakes and learning from mistakes are part of the process (Shulman 2005).

Move 5
Closure: Provide time for students to reflect individually and privately on what they have learned. Part of this move can be assignments to help students reinforce, elaborate, and extend understanding. Mathematical understanding develops over time through articulated and coherent mathematical struggles negotiated in small steps. These five teaching moves are exemplified using the chessboard reward task (Fey and Phillips 2005). See figure 1.

TACKLING THE TASK

Here we recount an enactment of the task in an inservice mathematics teachers’ professional development setting. The following vignette from a professional development instructor for secondary school mathematics teachers shows what it means to teach through problem solving.

Move 1
Pose the Chessboard Reward task. Allow teachers to make sense of the task (see fig. 1).

Fig. 1 This problem, used in a professional development setting, serves as the vehicle for the discussion of teaching moves.

Chessboard Reward Task
A king and queen want to reward a faithful peasant. The king suggests placing 1 ruba on the first square of a chessboard, 2 rubas on the second square, 4 rubas on the third, 8 on the fourth, and so on until all 64 squares are covered. The queen suggests placing 10 rubas on the first square, 25 on the second square, 40 on the third square, 55 on the fourth square, and so on. The servant receives the amount of money on the last square. How many rubas will be placed on the last square of the chessboard under each plan? Which plan is the better plan and why?

Adapted from Fey and Phillips (2005)

Move 2
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Move 2
Allow time for individual and interpersonal explorations. Teachers spent up to thirty minutes generating meaningful connected representations such as a table, or a grid to indicate payments.

Teachers generated and recorded the data into tables and computed differences, as shown in tables 1 and 2. The differences represent \( y_n - y_{n-1} \), \( n > 1 \). The professional development instructor emphasized the added need to consider \( x_n - x_{n-1} \), because the average rate of change is

\[
\Delta y = \frac{y_n - y_{n-1}}{x_n - x_{n-1}},
\]

and not just \( \Delta y \). As an example, some high school students run the risk of concluding, by focusing only on \( \Delta y \), that the relation given by the ordered pairs \{1, 2\}, \{4, 4\}, \{5, 6\}, \{20, 8\}, \{22, 10\} is linear.

Move 3
The teachers discussed the differences between the king’s and queen’s payment data. In the queen’s data, they observed that the first rate of change (difference) is a nonzero constant. Hence, they conjectured that the queen’s data are linear. The conjecture was tested by considering the constancy (predicated on similar right triangles) of slopes of lines. We also used the formula for computing the slope of lines, using an arbitrary ordered pair \((x, y)\) and \((1, 10)\) to generate the equation of the line:

\[
\frac{y - 10}{x - 1} = 1 \Rightarrow y = 10 - 15(x - 1) \Rightarrow y = 15x - 5
\]

The professional development instructor and the teachers explored the context significance of the \( y \)-intercept with the following question: What form of the equation of a line is the rule \( y = 15x - 5 \)? The discussion brought to the surface a potential misconception of the \( y \)-intercept as the starting point. After considering the practical domain of the task, the teachers concluded that graphically the \( y \)-intercept at \((0, -5)\) did not make sense in the problem context. However, as a formula, the rule tells us that the queen’s payments are \( 5 \) fewer than integer multiples of \( 15 \). The authors encourage the reader to consider the question: What other connections do you see arising from such an exploration?

Move 4
The constancy of the first rate of change of the queen’s payments contrasts with the rates of change of the king’s payments. The teachers observed that at each stage the rates of change of the king’s data are self-repeating and that the rates of change are exponential. They also argued that there would not be a last column in the finite difference table as they consider subsequent differences in the king’s data. Hence, they concluded that the notion of finite differences does not apply to exponential relations because it appears that there is not a finite number of steps that can yield a nonzero constant rate of change.

The teachers came up with a general rule for the \( n \)-th day payment by the king as \( 2^{(n-1)} \), where \( n \) represents a day number between 1 and 64. The rule emerged after serious discussions and debates, questioning peers, mathematical justifications, and mathematical representations. Some teachers used a scatter plot of the data to reconcile visually the growth pattern of the king’s payments. They used a graphing calculator (TI 84: ExpReg) to compute an exponential regression equation. The regression equation was \( f(x) = 0.5(2^x) \).
At this juncture, the professional development instructor asked: “Explain why—or why not—the two expressions $2^{x-1}$ and $0.5(2^x)$ are equivalent.” This question led to a consideration of properties of powers. The teachers shaped and molded the discourse, with minor direction from the professional development instructor. The instructor recorded the teachers’ work on the chalkboard and chart paper as an explicit and public record of the mathematics (Hiebert and Grouws 2007) intended to promote visibility and accountability (Shulman 2005).

A possible extension could be an examination of rates of change of the independent quantity over smaller equal intervals of the dependent quantity. One can begin the discussion via slope:

$$\frac{f(x + 0.1) - f(x - 0.1)}{(x + 0.1) - (x - 0.1)}$$

(see Coxford et al. 2001). The intent of such discussion is to forestall potential misconceptions about rates of change of exponential functions. Moreover, the extension can proceed into a discussion of limits and the derivative of the general family of functions $y = h^x$—here, $x \in \mathbb{R}$.

Discussions about the better payment plan between the queen and the king also brought out additional mathematical reasoning.

**Move 5**

Teachers chose the king’s model if they put themselves in the peasant’s shoes and considered an employment period longer than seven days. However, when the employment period is within a one-week period, the queen’s reward system became the plan of choice for the receiver of the remuneration.

Other intended mathematical residues emerged from the discussion of the prompt, “What is the mathematics embedded in the task?” These new insights related to the following:

- Rates of change of linear versus exponential relations
- Multiple connected and meaningful representations (verbal, equations, tables, including those for finite differences, and graphs)
- Mathematical practices of problem solving, reasoning, and communicating mathematical reasoning and mathematical thought

A caveat is necessary here: The rate of change was computed on the sequences as shown in

**tables 1 and 2. In table 2, the rates of change seem to imply that $y' = y$; this would have been the case if the base were $e$. Another potential incorrect overgeneralization could be a claim that the rates of change of families of functions reside in that family of functions. Such was the case for the functions used to model the chessboard reward task. However, the derivatives of functions do not have to be of the same family as the function. That is, $f''$ does not have to be in the same family of functions as $f$. A case in point is the logarithmic functions $\log x$, whose derivative is an algebraic function.**

**EXAMINING THE PEDAGOGY**

After the conclusion of the discussion of the mathematics and after having answered the questions of the task, the teachers and the professional development instructor turned their attention to understanding the pedagogy that was used. The instructor used this moment of reflecting on the pedagogy to make public the pedagogy of TtPS. The instructor also wanted to make the argument that TtPS is a robust approach to achieving equity in the classroom and that it can achieve depth of understanding without sacrificing breadth (that is, the intended curriculum can still be covered).

Basically, explanation of the pedagogy of TtPS involved stepping through the teaching moves as delineated in the previous section. Each move was made public by capturing its name and purpose on chart paper, which was displayed in a sequence that exhibited a pedagogical storyline of TtPS.

The aforementioned vignette from a professional development instructor for secondary school mathematics teachers shows what it means to teach through problem solving. The instructor used TtPS as a way to exemplify to teachers that TtPS should not be an activity “added on” to the beginning or the end of the unit. TtPS is a means of teaching important content.

Teaching important content in this way gives every student in every teacher’s classroom a chance to work with the mathematics instead of watching “experts” do all the work. The idea of letting each student struggle—productively—through mathematics and engage in mathematical practices is the meaning of the Equity Principle as outlined by NCTM. It is also a responsible way to ensure that students experience the joy, complexity, and beauty of mathematics. Let’s inspire students with the love of mathematics and invite them to partake in mathematical practices.
BIBLIOGRAPHY


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A fuller account of the five pedagogical moves of Teaching through Problem Solving is available by downloading one of the free apps for your smartphone and then scanning this tag to access www.nctm.org/mt012.